SOLUTION OF INTEGRAL EQUATIONS OF INVERSE PROBLEMS OF HEAT CONDUCTION

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UDC 536.24.02

A solution is given of integral equations of inverse problems of heat conduction by the method of successive approximations and also by means of expansions in orthogonal systems of functions.

It is a well-known fact that the solution of the heat-conduction equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

satisfying a zero initial condition for the half-line $x \geq 0$ and also the boundary condition

$$u(0,\ t) = \varphi(t), \tag{2}$$

is given by the integral

$$u(x,\ t) = \int_0^t \frac{x \exp\left[-\frac{x^2}{4a^2(t-\tau)}\right] \varphi(\tau)\, d\tau}{2a\sqrt{\pi(t-\tau)^3}}. \tag{3}$$

In the special case when $u(0,\ t) = u_0$, where u_0 is a constant, we can transform the integral (3), upon making the substitution $\alpha = x/2a\sqrt{t-\tau}$, to the form

$$v = \frac{2u_0}{\sqrt{\pi}} \int_{\frac{x}{2a\sqrt{t}}}^{\infty} \exp[-\alpha^2]\, d\alpha. \tag{4}$$

It is obvious that $v = u_0$ at $x = 0$, and also at $t = \infty$, since the integral attains its largest value, namely, $\sqrt{\pi}/2$, in the limit. If $u_0 = 1$, then $v < 1$ for $0 < t < \infty$.

With the aid of the solution (3) we can pose the inverse problem: Assuming an initial temperature of zero, we may ask what source $\varphi(t)$ at the point $x = 0$ will give rise, at some point $x = \xi$ of the half-line $x \geq 0$, to a specified temperature $f(t)$. Then, in accordance with Eq. (3), the inverse problem reduces to that of solving the following Volterra integral equation of the first kind:

$$\int_0^t \frac{\xi \exp\left[-\frac{\xi^2}{4a^2(t-\tau)}\right] \varphi(\tau)\, d\tau}{2a\sqrt{\pi(t-\tau)^3}} = f(t), \tag{5}$$

where the unknown function $\varphi(\tau)$ is determined as the solution of this equation. We have the inverse problem of heat conduction when we determine for the known value $f(t)$ the boundary condition $u(0,\ t) = \varphi(t)$. We substitute the value $\varphi(t)$ so determined into the expression (3) and obtain the temperature distribution in a semi-infinite rod.

We can pose the following more general problem: Let it be required to find the source $u(0,\ t) = \varphi(t)$, which would give rise to a temperature at a point $x = \xi$ of the half-line $x > 0$, differing from the temperature $\varphi(t)$ by a value equal to a specified function $f(t)$. Then, in accordance with the relation (3), we have a Volterra integral equation of the second kind,

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Voronezh State University. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 29, No. 1, pp. 120–123, July, 1975. Original article submitted February 18, 1975.

$$\varphi(t)=\lambda\int_0^t \frac{\xi \exp\left[-\frac{\xi^2}{4a^2(t-\tau)}\right]\varphi(\tau)\,d\tau}{2a\sqrt{\pi(t-\tau)^3}}+f(t), \tag{6}$$

where λ is a parameter. In this case we also have an inverse heat-conduction problem, wherein we wish to determine the boundary condition $\varphi(t)$, defined as the solution of Eq. (6) for $\lambda = 1$. Problems which reduce to Eqs. (5) and (6) can be found in [1].

We now clarify for which parameter values λ solutions of Eq. (6) exist.

We represent a solution of this equation in the form of a series,

$$\varphi(t)=\varphi_0(t)+\lambda\varphi_1(t)+\ldots+\lambda^n\varphi_n(t)+\ldots \tag{7}$$

Substituting the series (7) into Eq. (6), we have the following relations:

$$\varphi_0(t)=f(t);\ldots;\ \varphi_n(t)=\int_0^t \frac{\xi \exp\left[-\frac{\xi^2}{4a^2(t-\tau)}\right]\varphi_{n-1}(\tau)\,d\tau}{2a\sqrt{\pi(t-\tau)^3}}. \tag{8}$$

Assuming that $|f(t)| \le f_0$ for $0 \le t \le t_0$, we then have

$$|\varphi_1(t)| \leqslant f_0\int_0^t \frac{\xi \exp\left[-\frac{\xi^2}{4a^2(t-\tau)}\right]d\tau}{2a\sqrt{\pi(t-\tau)^3}} < f_0 v,$$

where $v < 1$ according to the expression (4).

Carrying out successive estimates of the expressions in Eq. (8), we obtain

$$|\varphi_n(t)| < f_0 v^n .$$

From this it follows that the terms of the unknown series (7) do not exceed in absolute value the terms of the majorizing series

$$f_0(1+\lambda v+\ldots+\lambda^n v^n+\ldots),$$

converging for any value $|\lambda| < 1/v$, including, in particular, the value $\lambda = 1$, since $v < 1$ for an arbitrary time t. Hence, the series (7) converges uniformly, and the function $\varphi(t)$ satisfies the integral equation (6), defining thereby a solution of the inverse problem. If in the series (7) we retain only the first several terms, we obtain an approximate value of the function $\varphi(t)$ with the help of successive approximations.

It should be noted that Eq. (5) becomes a Volterra equation of the second kind if we differentiate it through with respect to t, assuming for this that the derivative $f'(t)$ is continuous. The solution of the equation so obtained can be obtained by the same method used to solve Eq. (6).

We consider now the integral equation

$$F(x,t)=\int_0^t\int_a^b K(x,\xi,t-\tau)f(\xi,\tau)\,d\xi\,d\tau, \tag{9}$$

where the kernel

$$K(x,\xi,t-\tau)=\sum_{n=0}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)}{\lambda_n}\exp[-\lambda_n(t-\tau)] \tag{10}$$

is represented in a bilinear form in x and ξ with the aid of the orthonormal system of functions $\{\varphi_n(x)\}$ on the interval $[a, b]$. Here $F(x, t)$ is a known function; $f(x, t)$ is unknown; consequently, we have an integral equation of the first kind of mixed type (i.e., it is a Volterra equation with respect to the variable τ and a Fredholm equation with respect to the variable ξ).

We expand the function

$$F(x,t)=\sum_{n=0}^{\infty}F_n(t)\varphi_n(x) \tag{11}$$

in a series, where $\varphi_n(x)$ satisfies certain boundary conditions for $x = a$ and $x = b$. The coefficients $F_n(t)$

are given by

$$F_n(t) = \int_a^b F(\xi, t)\,\varphi_n(\xi)\,d\xi. \tag{12}$$

We also make the expansion

$$f(x, t) = \sum_{n=0}^{\infty} f_n(t)\,\varphi_n(x), \tag{13}$$

where the unknown coefficients $f_n(t)$ may be written, respectively,

$$f_n(t) = \int_a^b f(\xi, t)\,\varphi_n(\xi)\,d\xi. \tag{14}$$

Substituting the kernel (10) into Eq. (9), and also the series (13) and (14), we can write the equation

$$\sum_{n=0}^{\infty} F_n(t)\,\varphi_n(x) = \sum_{n=0}^{\infty} \int_0^t \left(\int_a^b f(\xi, \tau)\,\varphi_n(\xi)\,d\xi\right) \exp\left[-\lambda_n(t-\tau)\right] d\tau\,\varphi_n(x), \tag{15}$$

from which it follows that

$$F_n(t) = \int_0^t f_n(\tau) \exp\left[-\lambda_n(t-\tau)\right] d\tau. \tag{16}$$

But we can readily verify that

$$F_n'(t) + \lambda_n F_n(t) = f_n(t), \quad F_n(0) = 0. \tag{17}$$

Consequently, the coefficients $f_n(t)$ are determined in terms of the known coefficients $F_n(t)$; hence, upon substituting the $f_n(t)$ so determined into Eq. (13), we obtain the solution f(x, t) of Eq. (9) in the form of an expansion in terms of an orthogonal system of functions. By way of example, let us take the kernel of Eq. (9) in the form given in [2]:

$$K(x, \xi, t-\tau) = \frac{2}{l} \sum_{n=1}^{\infty} \sin\frac{n\pi}{l} x \sin\frac{n\pi}{l} \xi \exp\left[-\left(\frac{n\pi}{l}\right)^2 a^2 (t-\tau)\right]. \tag{18}$$

Using it, we shall solve the inverse problem for the nonhomogeneous heat-conduction equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x, t) \tag{19}$$

with the initial condition

$$u(x, 0) = 0 \tag{20}$$

and the boundary conditions

$$u(0, t) = 0, \quad u(l, t) = 0. \tag{21}$$

Then, in accordance with the expansions (11) and (13), we have

$$u(x, t) = \sum_{n=1}^{\infty} u_n(t) \sin\frac{n\pi}{l} x,$$

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t) \sin\frac{n\pi}{l} x,$$

where

$$u_n(t) = \frac{2}{l} \int_0^l u(\xi, t) \sin\frac{n\pi}{l} \xi\,d\xi,$$

$$f_n(t) = u_n'(t) + \left(\frac{n\pi}{l}\right)^2 a^2 u_n(t).$$

Thus, if the function u(x, t) is known, the solution of the inverse problem is determined by the function f(x, t).

Solutions of Eqs. (5) and (9), requiring special differentiability properties of the functions, can be found by the method of regularization.

The methods given here for solving integral equations of inverse heat conduction problems may be extended to multidimensional equations. Inverse problems were treated in [3] by a somewhat different method employing integral transformations.

LITERATURE CITED

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